

Radiation of charged particles by charged black hole

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Abstract

The probability of a charged particle production by the electric field of a charged black hole depends essentially on the particle energy. This probability is found in the nonrelativistic and ultrarelativistic limits. The range of values for the mass and charge of a black hole is indicated where the discussed mechanism of radiation is dominating over the Hawking one.

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1. The problem of particle production by the electric field of a black hole has been discussed repeatedly [1-6]. The probability of this process was estimated in these papers using in some way or another the result obtained previously [7-9] for the case of an electric field constant all over the space. This approximation might look quite natural with regard to sufficiently large black holes, for which the gravitational radius exceeds essentially the Compton wave length of the particle $\lambda = 1/m$. (We use in the present paper the units with $\hbar = 1$, $c = 1$; the Newton gravitational constant k is written down explicitly.) However, in fact, as will be demonstrated below, the constant-field approximation, generally speaking, is inadequate to the present problem, does not reflect a number of its essential peculiarities.

It is convenient to start the discussion just from the problem of particle creation by a constant electric field. Here and below we restrict to the consideration of the production of electrons and positrons, first of all because the probability of emitting these lightest charged particles is the maximum one. Besides, the picture of the Dirac sea allows one in the case of fermions to manage without the second-quantization formalism, thus making the consideration most transparent.

To calculate the main, exponential dependence of the effect, it is sufficient to restrict to a simple approach due to [7] (see also the textbook [10]). In the potential $-eEz$ of a constant electric field E the usual Dirac gap (Fig. 1) tilts (see Fig. 2). As a result, a particle which had a negative energy in the absence of the field, can now tunnel through the gap (see the dashed line in Fig. 2) and go to infinity as a usual particle. The hole created in this way is nothing but antiparticle. An elementary calculation leads to the well-known result for the probability of particle creation:

$$W \sim \exp \left(-\frac{\pi m^2}{eE} \right). \quad (1)$$

This simple derivation explains clearly some important properties of the phenomenon. First of all, the action inside the barrier does not change under a shift of the dashed line in Fig. 2 up or down. Just due to it expression (1) is independent of the energy of created particles. Then, for the external field to be considered as a constant one, it should change weakly along the path inside the barrier. However, the length of this path is not directly related to the Compton wave length of the particle. In particular, for an arbitrary weak field the path inside the barrier becomes arbitrary long.

Thus, one may expect that the constant-field approximation is not, generally speaking, applicable to the problem of a charged black hole radiation, and that the probability of particle production in this problem is strongly energy-dependent. The explicit form of this dependence will be found below. We restrict in the present work to the case of a non-rotating black hole.

2. We start the solution of the problem with calculating the action inside the barrier. The metric of a charged black hole is well-known:

$$ds^2 = f dt^2 - f^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (2)$$

where

$$f = 1 - \frac{2kM}{r} + \frac{kQ^2}{r^2}, \quad (3)$$

M and Q being the mass and charge of the black hole, respectively. The equation for a particle 4-momentum in these coordinates is

$$f^{-1} \left(\epsilon - \frac{eQ}{r} \right)^2 - fp^2 - \frac{l^2}{r^2} = m^2. \quad (4)$$

Here ϵ and p are the energy and radial momentum of the particle. We assume that the particle charge e is of the same sign as the charge of the hole Q , ascribing the charge $-e$ to the antiparticle.

Clearly, the action inside the barrier is minimum for the vanishing orbital angular momentum l . It is rather evident therefore (and will be demonstrated in the next section explicitly) that after the summation over l just the s -state defines the exponential in the total probability of the process. So, we restrict for the moment to the case of a purely radial motion. The equation for the Dirac gap for $l = 0$ is

$$\epsilon_{\pm}(r) = \frac{eQ}{r} \pm m\sqrt{f}. \quad (5)$$

It is presented in Fig. 3. It is known [11] that at the horizon of a black hole, for $r = r_+ = kM + \sqrt{k^2M^2 - kQ^2}$, the gap vanishes. Then, with the increase of r the lower boundary of the gap $\epsilon_-(r)$ decreases monotonically, tending asymptotically to $-m$. The upper branch $\epsilon_+(r)$ at first, in general, increases, and then decreases, tending asymptotically to m .

It is clear from Fig. 3 that those particles of the Dirac see whose coordinate r exceeds the gravitational radius r_+ and whose energy ϵ belongs to the interval $\epsilon_-(r) > \epsilon > m$, tunnel through the gap to infinity. In other words, a black hole loses its charge due to the discussed effect, by emitting particles with the same sign of the charge e , as the sign of Q . Clearly, the phenomenon takes place only under the condition

$$\frac{eQ}{r_+} > m. \quad (6)$$

For an extreme black hole, with $Q^2 = kM^2$, the Dirac gap looks somewhat different (see Fig. 4): when Q^2 tends to kM^2 the location of the maximum of the curve $\epsilon_+(r)$ tends to r_+ , and the value of the maximum tends to eQ/r_+ . It is obvious however that the situation does not change qualitatively due to it. Thus, though an extreme black hole has zero Hawking temperature and, correspondingly, gives no thermal radiation, it still creates charged particles due to the discussed effect.

In the general case $Q^2 \leq kM^2$ the doubled action inside the barrier entering the exponential for the radiation probability is

$$\begin{aligned} 2S &= 2 \int_{r_1}^{r_2} dr |p(r, \epsilon)| \\ &= 2 \int_{r_1}^{r_2} \frac{dr r}{r^2 - 2kMr + kQ^2} \sqrt{-p_0^2 r^2 + 2(\epsilon eQ - km^2 M)r - (e^2 - km^2)Q^2}. \end{aligned} \quad (7)$$

Here $p_0 = \sqrt{\epsilon^2 - m^2}$ is the momentum of the emitted particle at infinity, and the turning points $r_{1,2}$ are as usual the roots of the quadratic polynomial under the radical; we are interested in the energy interval $m \leq \epsilon \leq eQ/r_+$. Of course, the integral can be found explicitly, though it demands somewhat tedious calculations. However, the result is sufficiently simple:

$$2S = 2\pi \frac{m^2}{(\epsilon + p_0) p_0} [eQ - (\epsilon - p_0) kM]. \quad (8)$$

Certainly, this expression, as distinct from the exponent in formula (1), depends on the energy quite essentially.

Let us note that the action inside the barrier does not vanish even for the limiting value of the energy $\epsilon_m = eQ/r_+$. For a nonextreme black hole it is clear already from Fig. 3. For an extreme black hole this fact is not as obvious. However, due to the singularity of $|p(r, \epsilon)|$, the action inside the barrier is finite for $\epsilon = \epsilon_m = eQ/r_+$ for an extreme black hole as well. In this case the exponential factor in the probability is

$$\exp \left(-\pi \frac{\sqrt{k}m}{e} kmM \right). \quad (9)$$

Due to the extreme smallness of the ratio

$$\frac{\sqrt{k}m}{e} \sim 10^{-21}, \quad (10)$$

the exponent here is large only for a very heavy black hole, with a mass M exceeding that of the Sun by more than 5 orders of magnitude. And since the total probability, integrated over energy, is dominated by the energy region $\epsilon \sim \epsilon_m$, the semiclassical approach is applicable in the case of extreme black holes only for these very heavy objects. Let us note also that for the particles emitted by an extreme black hole, the typical values of the ratio ϵ/m are very large:

$$\frac{\epsilon}{m} \sim \frac{\epsilon_m}{m} = \frac{eQ}{kmM} = \frac{e}{\sqrt{k}m} \sim 10^{21}.$$

In other words, an extreme black hole in any case radiates highly ultrarelativistic particles mainly.

Let us come back to nonextreme holes. In the nonrelativistic limit, when $eQ/r_+ \rightarrow m$ and, correspondingly, the particle velocity $v \rightarrow 0$, the exponential is of course very small:

$$\exp \left(-\frac{2\pi kmM}{v} \right). \quad (11)$$

Therefore, we will consider mainly the opposite, ultrarelativistic limit where the exponential is

$$\exp \left(-\pi \frac{m^2}{\epsilon^2} eQ \right). \quad (12)$$

Of course, here also the energies $\epsilon \sim \epsilon_m \sim eQ/kM$ are essential, so that the ultrarelativistic limit corresponds to the condition

$$eQ \gg kmM. \quad (13)$$

But then the semiclassical result (12) is applicable (i.e., the action inside the barrier is large) only under the condition

$$kmM \gg 1. \quad (14)$$

Let us note that this last condition means that the gravitational radius of the black hole ($r_+ \sim kM$) is much larger than the Compton wave length of the electron $1/m$. In other words, the result (12) refers to macroscopic black holes. Combining (13) with (14), we arrive at one more condition for the applicability of formula (12):

$$eQ \gg 1. \quad (15)$$

We will come back to this relationship below.

Let us note that in [4] the action inside the barrier was being calculated under the same assumptions as formula (12). However, the answer presented in [4], $2S = \pi m^2 r_+^2 / eQ$, is independent of energy at all (and corresponds to formula (1) which refers to the case of a constant electric field). I do not understand how such an answer could be obtained for the discussed integral in the general case $\epsilon \neq \epsilon_m$.

3. The obtained exponential is the probability that a particle approaching the turning point r_1 (see Figs. 3, 4) from the left, will tunnel through the potential barrier. One should recall that in the general case the position of the turning point depends not only on the particle energy ϵ , but on its orbital angular momentum l as well. The total number of particles with given ϵ and l , approaching a spherical surface of the radius r_1 in unit time, is equal to the product of the area of this surface

$$S = 4\pi r_1^2(\epsilon, l) \quad (16)$$

times the current density of the particles

$$j^r(\epsilon, l) = \frac{\rho}{\sqrt{g_{00}}} \frac{dr}{dt} \quad (17)$$

(see, e.g., [12], §90). The particle velocity is as usual

$$v^r = \frac{dr}{dt} = \frac{\partial \epsilon}{\partial p} \quad (18)$$

(the subscript r of the radial momentum p is again omitted). To obtain an explicit expression for the particle density ρ , we will use the semiclassical approximation (the conditions of its applicability for the region $r_+ \leq r \leq r_1$ will be discussed later). Let us note that the volume element of the phase space

$$2 \frac{dp_x dp_y dp_z dx dy dz}{(2\pi)^3} \quad (19)$$

is a scalar. (The factor 2 here is due as usual to two possible orientations of the electron spin.) On the other hand, the number of particles in the elementary cell $dx dy dz$ equals (see [12], §90)

$$\rho \sqrt{\gamma} dx dy dz, \quad (20)$$

where γ is the determinant of the space metric tensor. Since all the states of the Dirac sea are occupied, we obtain by comparing formulae (19) and (20) that the following expression

$$\frac{\rho}{\sqrt{g_{00}}} = \frac{2}{\sqrt{g_{00}\gamma}} \sum \frac{dp_x dp_y dp_z}{(2\pi)^3} = \frac{2}{\sqrt{-g}} \sum \frac{dp_x dp_y dp_z}{(2\pi)^3}$$

should be plugged in formula (17) for the current density (the summation here and below is performed with fixed ϵ and l , see (17)). In our case the determinant g of the four-dimensional metric tensor does not differ from the flat one, so that the radial current density of the particles of the Dirac sea is

$$j^r(\epsilon, l) = 2 \sum \frac{d^3 p}{(2\pi)^3} \frac{\partial \epsilon}{\partial p}. \quad (21)$$

The summation in the right-hand-side reduces in fact to the multiplication by the number $2l + 1$ of possible projections of the orbital angular momentum \mathbf{l} onto the z axis and to the integration over the azimuth angle of the vector \mathbf{l} , which gives 2π . With the account for the identity

$$\frac{\partial \epsilon}{\partial p_r} dp_r = d\epsilon,$$

we obtain in the result

$$j^r(\epsilon, l) = 2 \frac{2\pi(2l + 1)}{(2\pi)^3 r_1^2(\epsilon, l)}. \quad (22)$$

Finally, the pre-exponential factor in the probability, differential in energy and orbital angular momentum, is

$$\frac{2(2l + 1)}{\pi}. \quad (23)$$

Correspondingly, the number of particles emitted per unit time is

$$\frac{dN}{dt} = \frac{2}{\pi} \int d\epsilon \sum_l (2l + 1) \exp[-2S(\epsilon, l)]. \quad (24)$$

In the most interesting, ultrarelativistic case dN/dt can be calculated explicitly. Let us consider the expression for the momentum in the region inside the barrier for $l \neq 0$

$$|p(\epsilon, l, r)| = f^{-1} \sqrt{\left(m^2 + \frac{l^2}{r^2}\right) f - \left(\epsilon - \frac{eQ}{r}\right)^2}. \quad (25)$$

The main contribution to the integral over energies in formula (24) is given by the region $\epsilon \rightarrow \epsilon_m$. In this region the functions $f(r)$ and $\epsilon - eQ/r$, entering expression (25), are small and change rapidly. As to the quantity

$$\mu^2(r, l) = m^2 + \frac{l^2}{r^2}, \quad (26)$$

one can substitute in it for r its average value, which lies between the turning points r_1 and r_2 . Obviously, in the discussed limit $\epsilon \rightarrow \epsilon_m$ the near turning point coincides with the horizon radius, $r_1 = r_+$. And the expression for the distant turning point is in this limit

$$r_2 = r_+ \left[1 + \frac{2\mu^2}{\epsilon_m^2 - \mu^2} \frac{\sqrt{k^2 M^2 - kQ^2}}{r_+} \right]. \quad (27)$$

Assuming that for estimates one can put in formula (26) $r \sim r_+$, one can easily show that the correction to 1 in the square bracket is bounded by the ratio $l^2/(eQ)^2$. Assuming that this ratio is small (we will see below that this assumption is self-consistent), we arrive at the conclusion that $r_2 \approx r_+$, and hence μ^2 can be considered independent of r : $\mu^2(r, l) = m^2 + l^2/r_+^2$. As a result, we obtain

$$2S(\epsilon, l) \approx \pi eQ \left(\frac{m^2}{\epsilon^2} + \frac{l^2}{r_+^2 \epsilon^2} \right). \quad (28)$$

Now we find easily

$$\frac{dN}{dt} = m \left(\frac{eQ}{\pi m r_+} \right)^3 \exp \left(- \frac{\pi m^2 r_+^2}{eQ} \right). \quad (29)$$

Let us note that the range of orbital angular momenta, contributing to the total probability (29), is effectively bounded by the condition $l^2 \leq eQ$. Since $eQ \gg 1$, this condition allows one to change from the summation over l in formula (24) to the integration. On the other hand, this condition justifies the used approximation $\mu^2(r, l) = m^2 + l^2/r_+^2$.

However, up to now we have not considered one more condition necessary for the derivation of formula (29). We mean the applicability of the semiclassical approximation to the left of the barrier, for $r_+ \leq r \leq r_1$. This condition has the usual form

$$\frac{d}{dr} \frac{1}{p(r)} < 1. \quad (30)$$

In other words, the minimum size of the initial wave packet should not exceed the distance from the horizon to the turning point. Using the estimate

$$p(r) \sim \frac{r_+ (eQ - \epsilon r_+)}{(r - r_+)(r - r_-)}$$

for the momentum in the most essential region, one can check that for an extreme black hole the condition (30) is valid due to the bound $eQ \gg 1$. In a non-extreme case, for $r_+ - r_- \sim r_+$, the situation is different: the condition (30) reduces to

$$\epsilon < \frac{eQ - 1}{r_+} \sim \frac{eQ}{r_+}. \quad (31)$$

Thus, for a non-extreme black hole in the most essential region $\epsilon \rightarrow \epsilon_m$ the condition of the semiclassical approximation is not valid. Nevertheless, the semiclassical result (24) remains true qualitatively, up to a numerical factor in the pre-exponential.

In concluding this section few words on the radiation of light charged black holes, for which $kmM < 1$, i.e., for which the gravitational radius is less than the Compton wave length of the electron. In this case the first part,

$$\epsilon < \frac{eQ - 1}{r_+},$$

of inequality (31), which guarantees the localization of the initial wave packet in the region of a strong field, means in particular that

$$eQ = Z\alpha > 1 \tag{32}$$

(we have introduced here $Z = Q/e$). It is well-known (see, e.g., [13, 14]) that the vacuum for a point-like charge with $Z\alpha > 1$ is unstable, so that such an object loses its charge by emitting charged particles. It is quite natural that for a black hole whose gravitational radius is smaller than the Compton wave length of the electron, the condition of emitting a charge is the same as in the pure quantum electrodynamics. (Let us note that the unity in all these conditions should not be taken too literally: even in quantum electrodynamics, where the instability condition for the vacuum of particles of spin 1/2 is for a point-like nucleus just $Z\alpha > 1$, for a finite-size nucleus it changes [13, 14] to $Z\alpha > 1.24$. On the other hand, for the vacuum of scalar particles in the field of a point-like nucleus the instability condition is [15, 16]: $Z\alpha > 1/2$.) As has been mentioned already, for a light black hole, with $kmM < 1$, the discussed condition $eQ > 1$ leads to a small action inside the barrier and to the inapplicability of the semiclassical approximation used in the present article. The problem of the radiation of a charged black hole with $kmM < 1$ was investigated numerically in [17].

4. The exponential

$$\exp\left(-\frac{\pi m^2 r_+^2}{eQ}\right)$$

in our formula (29) coincides with the expression arising from formula (1), which refers to a constant electric field E , if one plugs in for this field its value Q/r_+^2 at the black hole horizon. As has been mentioned already, an approach based on formulae for a constant electric field was used previously in Refs. [1-6]. Thus, our result for the main, exponential dependence of the probability integrated over energies, coincides with the corresponding result of these papers. Moreover, our final formula (24) agrees with the corresponding result of Ref. [6] up to an overall factor 1/2. (This difference is of no interest by itself: as has been noted above, for a non-extreme black hole the semiclassical approximation cannot guarantee at all an exact value of the overall numerical factor.)

Nevertheless, we believe that the analysis of the phenomenon performed in the present work, which demonstrates its essential distinctions from the particle production by a constant external field, is useful. First of all, it follows from this analysis that the probability of the particle production by a charged black hole has absolutely nontrivial energy spectrum. Then, in no way are real particles produced by a charged black hole all over the whole space: for

a given energy ϵ they are radiated by a spherical surface of the radius $r_2(\epsilon)$, this surface being close to the horizon for the maximum energy. (It follows from this, for instance, that the derivation of the mentioned result of Ref. [6] for dN/dt has no physical grounds: this derivation reduces to plugging $E = Q/r^2$ into the well-known Schwinger formula [9], obtained for a constant field, with subsequent integrating all over the space outside the horizon.)

Let us compare now the radiation intensity I due to the effect discussed, with the intensity I_H of the Hawking thermal radiation. Introducing additional weight ϵ in the integrand of formula (24), we obtain

$$I = \pi m^2 \left(\frac{eQ}{\pi m r_+} \right)^4 \exp \left(- \frac{\pi m^2 r_+^2}{eQ} \right). \quad (33)$$

As to the Hawking intensity, the simplest way to estimate it, is to use dimensional arguments, just to divide the Hawking temperature

$$T_H = \frac{1}{4\pi r_+}$$

by a typical classical time of the problem r_+ (in our units $c = 1$). Thus,

$$I_H \sim \frac{1}{4\pi r_+^2}. \quad (34)$$

More accurate answer for I_H differs from this estimate by a small numerical factor $\sim 2 \cdot 10^{-2}$, but for qualitative estimates one can neglect this distinction. The intensities (33) and (34) get equal for

$$eQ \sim \frac{\pi}{6} \frac{(mr_+)^2}{\ln(mr_+)} \sim \frac{\pi}{6} \frac{(kmM)^2}{\ln(kmM)}. \quad (35)$$

(One cannot agree with the condition $eQ \sim 1/(4\pi)$ for the equality of these intensities, derived in Ref. [6] from the comparison of $\epsilon_m = eQ/r_+$ with $T_H = 1/(4\pi r_+)$.)

Let us consider in conclusion the change of the horizon surface of a black hole, and hence of its entropy, due to the discussed non-thermal radiation. To this end, it is convenient to introduce, following Ref. [18], the so-called irreducible mass M_0 of a black hole:

$$2M_0 = M + \sqrt{M^2 - Q^2}; \quad (36)$$

here and below we put $k = 1$. This relationship can be conveniently rewritten also as

$$M = M_0 + \frac{Q^2}{4M_0}. \quad (37)$$

Obviously, $r_+ = 2M_0$, so that the horizon surface and the black hole entropy are proportional to M_0^2 .

When a charged particle is emitted, the charge of a black hole changes by $\Delta Q = -e$, and its mass by $\Delta M = -eQ/r_+ + \xi$, where ξ is the deviation of the particle energy from the maximum one. Using the relationship (37), one can easily see that as a result of the radiation,

the irreducible mass M_0 , and hence the horizon surface and entropy of a non-extreme black hole do not change if the particle energy is the maximum one eQ/r_+ . In other words, such a process, which is the most probable one, is adiabatic. For $\xi > 0$, the irreducible mass, horizon surface, and entropy increase.

As usual, an extreme black hole, with $M = Q = 2M_0$, is a special case. Here for the maximum energy of an emitted particle $\epsilon_m = e$, we have $\Delta M = \Delta Q = -e$, so that the black hole remains extreme after the radiation. In this case $\Delta M_0 = -e/2$, the irreducible mass and the horizon surface decrease. In a more general case, $\Delta M = -e + \xi$, the irreducible mass changes as follows:

$$\Delta M_0 = -\frac{e - \xi}{2} + \sqrt{\left(M_0 - \frac{e}{2} + \frac{\xi}{4}\right)\xi}. \quad (38)$$

Clearly, in the case of an extreme black hole of a large mass, already for a small deviation ξ of the emitted energy from the maximum one, the square root is dominating in this expression, so that the horizon surface increases.

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$+ m$

$- m$

Fig. 1

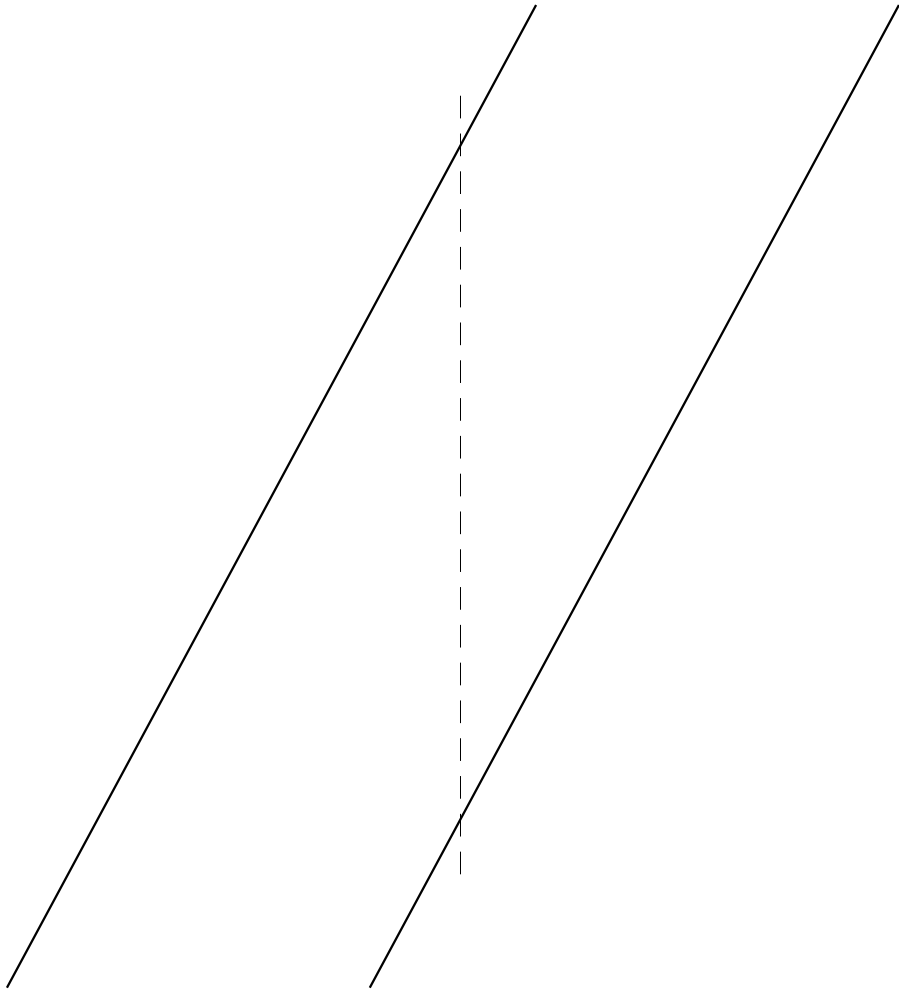


Fig. 2

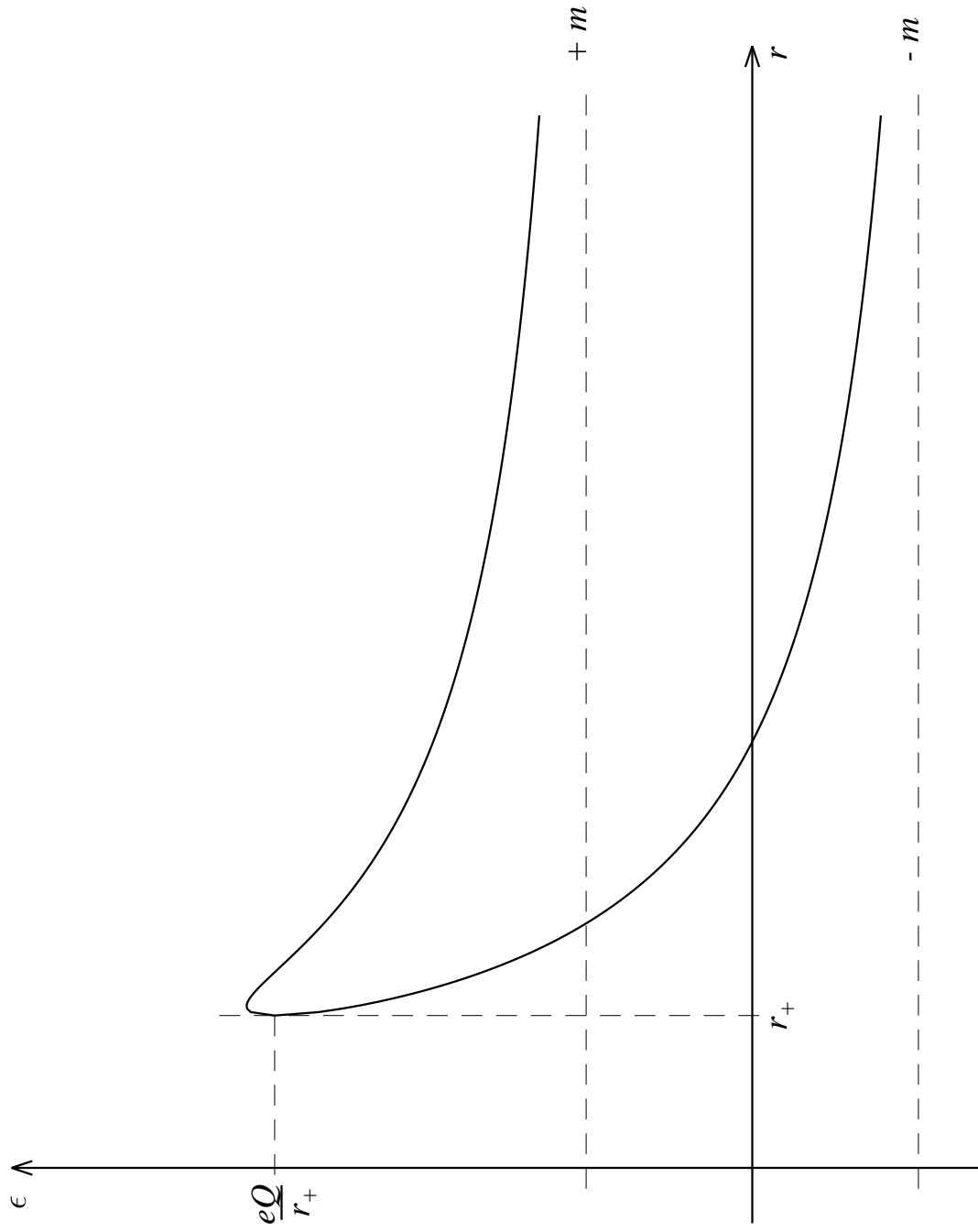


Fig. 3

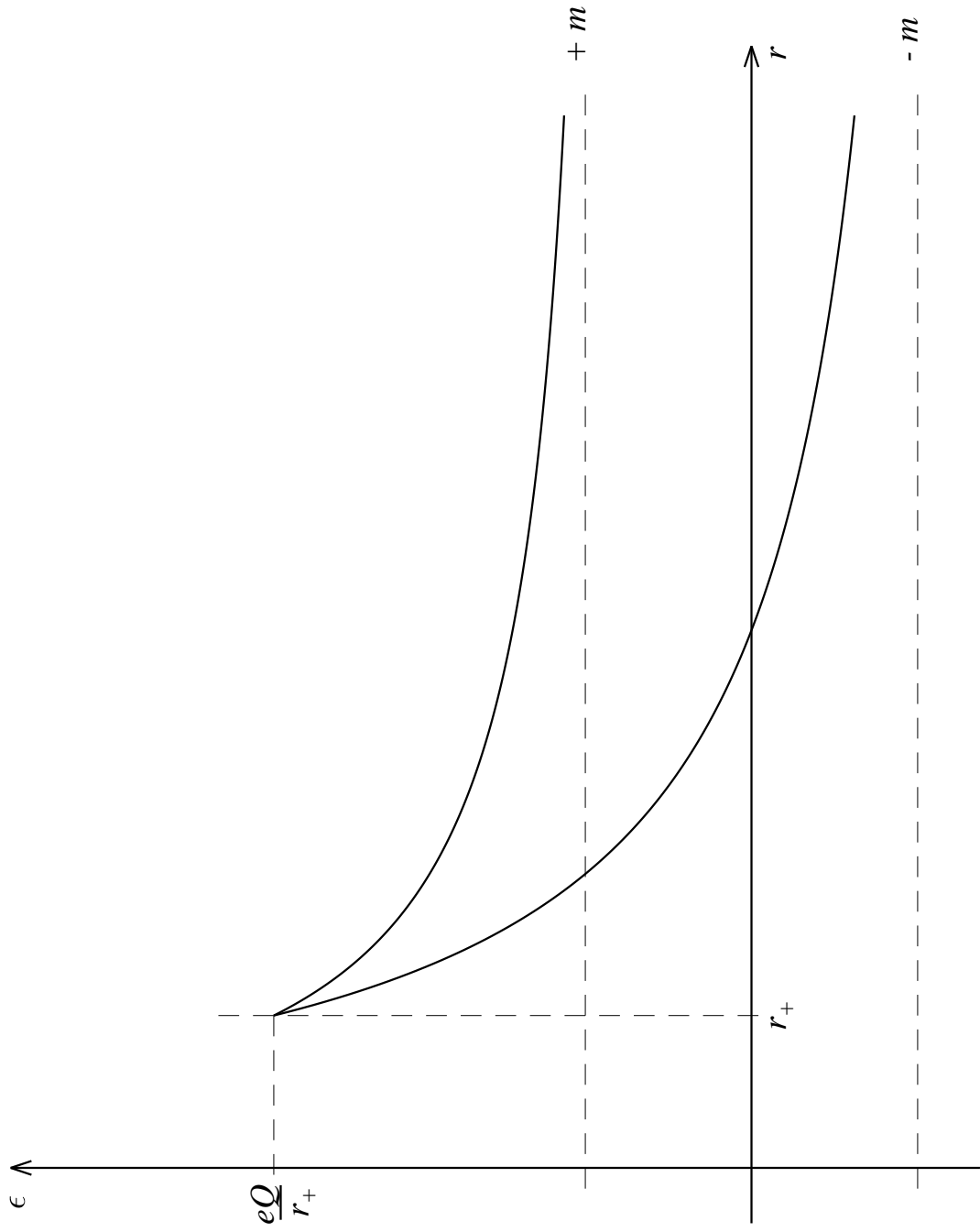


Fig. 4